Complex Plane: Vectors, Conjugates, and the Field Axioms

The Complex Plane as Vectors, Conjugation, Absolute Value, the Field Axioms, and the Matrix Form

Lecture Notes

September 8, 2026

1 Complex Numbers as Vectors

A complex number $z = a + ib$ can be mapped to a vector $\binom{a}{b}$ in the Cartesian plane $\mathbb{R}^2$.

Standard notation. The real part is $a = \mathrm{Re}(z)$ and the imaginary part $b = \mathrm{Im}(z)$. The complex plane $\mathbb{C}$ is $\mathbb{R}^2$ equipped with specific rules for addition and multiplication: the real line $\mathbb{R}$ is identified with vectors $\binom{a}{0}$ on the x-axis, while purely imaginary numbers lie along the y-axis.

Addition behaves exactly like standard vector addition: $(a + ib) + (c + id) = (a + c) + i(b + d)$, i.e. $\binom{a}{b} + \binom{c}{d} = \binom{a+c}{b+d}$.

Multiplication is defined algebraically as $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$, i.e.

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}, \qquad \text{so in particular } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

2 Absolute Value and Conjugate

The Complex Conjugate

For $z = a + ib$, the **complex conjugate** is $\bar{z} := a - ib$, the reflection with respect to the *real line*.

Component formulas

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i} = -\frac{i}{2}(z - \bar{z}).$$

Example (line equation). A line $ax + by = c$ (real notation, $a^2 + b^2 \neq 0$) becomes, substituting $x = \dfrac{z + \bar{z}}{2}$ and $y = \dfrac{z - \bar{z}}{2i}$,

$$a \cdot \frac{z + \bar{z}}{2} + b \cdot \frac{z - \bar{z}}{2i} = c, \qquad \text{i.e.} \qquad \frac{a - ib}{2}\, z + \frac{a + ib}{2}\, \bar{z} = c.$$

Writing $\mu := \dfrac{a - ib}{2}$ (so $\mu \neq 0$ and $\bar{\mu} = \dfrac{a + ib}{2}$), every line in $\mathbb{C}$ takes the compact form

$$\mu z + \bar{\mu}\bar{z} = c, \qquad \mu \neq 0,\ c \in \mathbb{R}.$$

Basic properties. $\bar{\bar{z}} = z$, $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, and (for $w \neq 0$) $\overline{z/w} = \bar{z}/\bar{w}$.

Absolute Value (Modulus)

$|z|^2 = x^2 + y^2 = (x+iy)(x-iy) = z\bar{z} = |\bar{z}|^2$. $|z|$ is the distance from z to 0.

Properties

Multiplicative: $|zw| = |z||w|$. *Proof:* $|zw|^2 = zw\overline{zw} = zw\bar{z}\bar{w} = z\bar{z}\,w\bar{w} = |z|^2|w|^2$.

Triangle inequality: $|z+w| \le |z|+|w|$. *Proof:* $|z+w|^2 = (z+w)(\overline{z+w}) = |z|^2+|w|^2+z\bar{w}+\bar{z}w = |z|^2+|w|^2+2\,\mathrm{Re}(z\bar{w})$. Since $\mathrm{Re}(z\bar{w}) \le |z\bar{w}| = |z||w|$, this is $\le |z|^2+|w|^2+2|z||w| = (|z|+|w|)^2$.

Scalar product

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$. Then we have the following complex expression for the *scalar product*

$$\left\langle \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right\rangle = a_1a_2 + b_1b_2 = \mathrm{Re}(z_1\overline{z_2}).$$

Set Notation

Open ball: $B(z,\delta) := \{w : |z-w| < \delta\}$. **Closed ball:** $\bar{B}(z,\delta) := \{w : |z-w| \le \delta\}$.

3 Complex Numbers Form a Field

Field Axioms

Here z_1, z_2, z_3 denote arbitrary elements of $\mathbb{C}$.

(P1) Associative addition: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

(P2) Additive identity: $z_1 + 0 = 0 + z_1 = z_1$.

(P3) Additive inverses: $z_1 + (-z_1) = (-z_1) + z_1 = 0$.

(P4) Commutative addition: $z_1 + z_2 = z_2 + z_1$.

(P5) Associative multiplication: $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$.

(P6) Multiplicative identity: $z_1 \cdot 1 = 1 \cdot z_1 = z_1$, provided $1 \neq 0$.

(P7) Multiplicative inverses: $z_1 \cdot z_1^{-1} = z_1^{-1} \cdot z_1 = 1$ for $z_1 \neq 0$.

(P8) Commutative multiplication: $z_1 \cdot z_2 = z_2 \cdot z_1$.

(P9) Distributive law: $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$.

Division rules. $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$ (equivalently, $z\bar{z} = |z|^2$), and $\dfrac{w}{z} = \dfrac{w\bar{z}}{|z|^2}$.

4 Matrix Form of a Complex Number

For a fixed $z = a + ib$, multiplying by $w = x + iy$ is a linear transformation $M_z\vec{w} := zw$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}, \qquad M_z := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Matrix addition: $M_{z_1} + M_{z_2} = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{pmatrix} = M_{z_1+z_2}$.

Matrix multiplication: $M_{z_1} \cdot M_{z_2} = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2 & -(a_2b_1 + a_1b_2) \\ a_2b_1 + a_1b_2 & a_1a_2 - b_1b_2 \end{pmatrix} = M_{z_1z_2}$.

Theorem (field isomorphism)

Let $\mathcal{M} := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$. The map $\phi : \mathbb{C} \to \mathcal{M}$, $z \mapsto M_z$, is a field isomorphism – a bijection preserving both addition and multiplication.

Proof. The entries of M_z recover $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$, so ϕ is injective; it is surjective onto $\mathcal{M}$ by definition of $\mathcal{M}$. Hence ϕ is a bijection. Additivity $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2)$ and multiplicativity $\phi(z_1z_2) = \phi(z_1)\phi(z_2)$ are exactly the matrix identities $M_{z_1} + M_{z_2} = M_{z_1+z_2}$ and $M_{z_1} \cdot M_{z_2} = M_{z_1z_2}$ established above. $\square$

The conjugate matrix. $M_z^T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = M_{\bar{z}}$.